Evaluating the Greeks for the LIBOR Market Model using the Pathwise Adjoint on GPU

Real-time hedging interest rates

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Agenda

1. Context
2. LIBOR Market model & the Greeks
3. Solution
4. Computational Experiments
5. Conclusion
A bit of terminology You must understand!

- **LIBOR** – the average interest rate estimated by leading banks that they would be charged if borrowing from other banks
- **Option** – a contract that gives an opportunity to buy or sell the asset in the future at a fixed price negotiated ”now”
- **Hedging** - the method to compensate loss (gains) on the market by gains (loss) on the option market
- **Interest Rate** – the rate at which interests are paid a lender by a borrower
- **Forward Interest Rate** - the interest rate specified for a loan that will occur in the future
- **the Greeks** – measure how model parameters (observable on the market) affect the future predicted payoff

**Question:** How to model the behaviour of LIBOR and predict Forward Interest Rates?
How does LMM work? Why important?

Borrower 

a 1 billion $ credit for 1 year

Lender

1 billion $ + interests dependent on LIBOR after 1 year
How does LMM work? Why important?

How to mitigate credit risk (LIBOR movements)?

- a 1 billion $ credit for 1 year
- 1 billion $ + interests dependent on LIBOR after 1 year
How does LMM work? Why important?

- **Borrower**
  - Options (caplets) on interest rates (LIBOR) with different maturities (up to 1 year)

- **Lender**
  - A $1 billion credit for 1 year
  - $1 billion + interests dependent on LIBOR after 1 year

- **Option Market**

The diagram illustrates the concept of LMM (Local Linear Model) in the context of financial derivatives and credit lines. It shows the interaction between a borrower, a lender, and the option market, highlighting the key elements of credit lines and the role of interest rate derivatives.
How does LMM work? Why important?

- How to hedge a $1 billion credit?
- a $1 billion credit for 1 year
- 1 billion + interest after
- How to price options on interest rates?
- options (caplets) on interest rates (LIBOR) with different maturities (up to 1 year)

Borrower

Option Market
How does LMM work? Why important?

The LIBOR Market Model

Borrower

1 billion $ + interest

options (caplets) on interest rates (LIBOR) with different maturities (up to 1 year)

A 1 billion $ credit for 1 year
How does LMM work? Why important?

I need to calculate the LMM Greeks to rebalance my portfolio and minimize interests or even make profits.

a 1 billion $ credit for 1 year

1 billion $ + interest

I need to use the LIBOR market model to price options.

options (caplets) on interest rates (LIBOR) with different maturities (up to 1 year)

Borrower

Option Market
How does LMM work? Why important?

I will use the LMM + the Adjoint + HPC to calculate the Greeks

Borrower

I will use the LMM + the Adjoint + HPC to price options

Option Market

1 billion $ credit for 1 year

1 billion $ + interest after

options (caplets) on interest rates (LIBOR) with different maturities (up to 1 year)
Why important? Contribution

Why important? More?

Many financial models are dependent on the LIBOR market model:

- European interest rate products: swaps/swaptions
- Bermudan interest rate products: swaps/swaptions
- Exotic interest rate products: Snowball swaps
- Cross currency exotic interest rate products

LIBOR settled at 1d, 1w, 2w, 1m, 2m ..., 12m

Total OTC interest rate derivatives turnover: 2.3 trillion $ (2013)

This work can be applied to hedge and price above options!!!

Contribution

- Most commercial products are sequential and use slow pathwise or inaccurate Finite Difference methods

- ..., but this must be accurate and work as fast as possible. How?

- Solution: The LIBOR Market Model + the Adjoint + GPU
Given a set of dates \( T_1, T_2, \ldots, T_{N-1}, T_N \) let:

\[
L_i(T_n), \ i > n, \ n = 1, 2, \ldots, N - 1, N
\]

be the forward interest rate at time \( T_n \) for the period \( dt_i = T_{i+1} - T_i \)

<table>
<thead>
<tr>
<th>( dt_i )</th>
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How to predict forward interest rates for different periods in the future?
Let $L^n_i$ denote the forward LIBOR rate for the time $dt_i = T_{i+1} - T_i$ at time $n$ (and $i > n$) then the evolution of the forward rates $L^n_i$ for $n = 0, \ldots, N_{mat} - 1$ is expressed as:

$$L^{n+1}_i = L^n_i \exp\left(\left(\sigma_{i-n-1}S^n_i - \frac{1}{2}\sigma_{i-n-1}^2\right)dt_i + \sigma_{i-n-1}Z^n\sqrt{(dt_i)}, i > n\right)$$  \hspace{1cm} (2)$$

where $Z^n$ is the random number with normal distribution, and $\sigma_{i-n-1}$ is the interest rate volatility at time $i$ and $S^n_i$ is:

$$S^n_i = \sum_{j=n+1}^{i} \frac{\sigma_{j-n-1}dt_iL^n_j}{1 + dt_iL^n_j}, i > n$$  \hspace{1cm} (3)$$
Expected $\bar{L}_i^N$ is estimated using the average of independent replications of $L_{ij}^N$ with different $Z^n$

$$
\bar{L}_i^N = \mathbb{E}(L_i^N) \approx \frac{\sum_{j=0}^{\text{Num paths}} L_{ij}^N}{\text{Num paths}}
$$

(4)

$j$ - is the index representing the j-th replication (a scenario of Monte-Carlo simulation)
Caplets - European options on interest rates

Definition
Caplet is an option in which the buyer receives payment when the interest rate exceeds the agreed strike price. Its payoff is:

\[ p_i = N \frac{period}{1\text{year}} \max(L_i - K_i, 0) \]  \hspace{1cm} (5)

where \( N \) is the notional of money the buyer invests

Example
Suppose You own a caplet on the year US interest rate: 2.5 % that expires tomorrow. If tomorrow’s interest rate is 3 % and You have $ 1 million then You will gain:

\[ \$ 1 \text{ million} \max(0.03 - 0.025, 0) = \$ 2500 \]
The Greeks for the Libor Market Model

The Greeks (risk sensitivities)

- The first-order/second-order derivatives of the payoff with respect to model parameters
- They measure how model parameters affect the future payoff
- Traders hedge their portfolio – compensate loss (gain) on the market by gain (loss) made on the derivatives market
- the Greeks need to be constant over an investment period, but market prices fluctuate...
- ... thus, traders frequently rebalance their portfolio

For example, consider the Greek delta for the i-th period (with respect to the initial forward rate $L_i(T_0)$):

$$\frac{dp_i(T_N)}{dL_i(T_0)} = \frac{\partial p_i(T_N)}{\partial L_i(T_N)} \frac{\partial L_i(T_N)}{\partial L_i(T_{N-1})} \frac{\partial L_i(T_{N-1})}{\partial L_i(T_{N-2})} \cdots \frac{\partial L_i(T_2)}{\partial L_i(T_1)} \frac{\partial L_i(T_1)}{\partial L_i(T_0)}$$

(6)
The Greeks for the Libor Market Model

For the LMM model we need to calculate all the Greeks for each period (each predicted forward interest rate)

\[
\frac{dp_i(T_N)}{dL_i(T_0)} = \frac{\partial p_i(T_N)}{\partial L_i(T_N)} \frac{\partial L_i(T_N)}{\partial L_i(T_{N-1})} \frac{\partial L_i(T_{N-1})}{\partial L_i(T_{N-2})} \ldots \frac{\partial L_i(T_2)}{\partial L_i(T_1)} \frac{\partial L_i(T_1)}{\partial L_i(T_0)}
\]  

(7)

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<th>(T_1)</th>
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<th>(T_{N-2})</th>
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<td>(\frac{\partial L_2(T_2)}{\partial L_2(T_0)})</td>
<td>(\frac{\partial L_3(T_N)}{\partial L_3(T_0)})</td>
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<td>(\frac{\partial L_{N-1}(T_{N-1})}{\partial L_{N-1}(T_0)})</td>
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<td>(\frac{\partial L_N(T_N)}{\partial L_N(T_0)})</td>
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</table>

The Greeks with respect to all the volatilities \(\sigma^n_i\) are also crucial.

**All this work must be done for all the paths!!!
Remember $L^n_i$ denotes the forward LIBOR rate for the time $dt_i = T_{i+1} - T_i$ at time $n$ and $i > n$:

$$L^{n+1}_i = L^n_i \exp\left((\sigma_{i-n-1}S^n_i - \frac{1}{2}\sigma^2_{i-n-1})dt_i + \sigma_{i-n-1}Z^n\sqrt{dt_i}\right), i > n \quad (8)$$

If $S_i$ for the $i$-th period is:

$$S^n_i = \sum_{j=n+1}^{i} \frac{\sigma_{j-n-1}dt_i L^n_j}{1 + dt_i L^n_j}, i > n \quad (9)$$

then $S_{i+1}$ for the $(i+1)$-th period is:

$$S^n_{i+1} = S^n_i + \frac{\sigma_{i-n}dt_i L^n_{i+1}}{1 + dt_i L^n_{i+1}}, i > n \quad (10)$$

- Recursive dependency important in an efficient implementation
Therefore, to calculate the forward interest rate $L_{i+1}(T_k)$ at time $T_k$ for the $(i + 1)$-th period, we should use the previously calculated $S_{i}^{k}$ (from $L_{i}(T_k)$) and add some additional factor:

$$\frac{\sigma_{i-n}dt_i L_{i+1}^{k}}{1 + dt_i L_{i+1}^{k}}$$

(11)

Note, this will need synchronization if the forward interest rates at time $k$ for different periods are calculated in parallel.
Solution – the LMM Greeks and the Adjoint on a CPU

- All the partial derivatives: \( \frac{\partial L_i(T_{k+1})}{\partial L_i(T_k)} \), \( \frac{\partial S_i(T_k)}{\partial L_i(T_k)} \), \( \frac{\partial L_i(T_{k+1})}{\partial \sigma_i^k} \), \( \frac{\partial S_i(T_k)}{\partial \sigma_i^k} \) are evaluated by the Adjoint.
- Through the MC evolution the partial results are recursively updated to produce the final Greeks.

Evolution of a single scenario
(Monte-Carlo simulation)

<table>
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<tr>
<th>( T_1 )</th>
<th>( T_2 )</th>
<th>( T_3 )</th>
<th>...</th>
<th>( T_{N-2} )</th>
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The Adjoint
Solution – the LMM Greeks and the Adjoint on a GPU

- Each thread calculates the forward interest rate for the $i$-th period (Each ”row” is processed by a single thread)
- Each thread block calculates a MC scenario with different tenor dates

![Evolution of a single scenario](image)

- The Adjoint
- Adding each $S^n$ needs synchronization within threads
Computational Experiments

- CPU execution times & Speedup - The first-order LMM Greeks (N = 1024 different periods, forward interest rates)

<table>
<thead>
<tr>
<th>Number of paths</th>
<th>CPU time (seconds)</th>
<th>CUDA (2050)</th>
<th>OpenCL (2050)</th>
<th>CUDA (K40)</th>
<th>OpenCL (K40)</th>
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<td>27</td>
<td>42x</td>
<td>41x</td>
<td>49x</td>
<td>47x</td>
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<td>44x</td>
<td>43x</td>
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<td>44x</td>
<td>43x</td>
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<tr>
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<tr>
<td>1000</td>
<td>272</td>
<td>45x</td>
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Test Environment:
- CPU: an Intel Xeon Sandybridge X5650 2.66 Ghz (16 cores)
- CUDA and OpenCL: an NVIDIA Tesla M2070 (448 cores) and an NVIDIA K40c (2880 cores)

Both parallel and sequential libraries were compiled with the optimization commands

```
--use_fast_math --ptxas-options=-v,-03
```
Computational Experiments

- Speedup - The first-order LMM Greeks
  \((N = 1024 \text{ different periods, forward interest rates})\)
Monte Carlo simulation + GPU + Pathwise Adjoint for the LIBOR market model:

- increases performance of Monte-Carlo simulation for the LIBOR market model
- is much faster than any other numerical methods (pathwise, FD, likelihood) – see, the LMM work of Prof. Mike Giles (University of Oxford)
- reduces a computational effort of the Greeks’ calculation by three orders of magnitude
- improves accuracy of the Greeks in comparison to pathwise, FD or likelihood methods (a single Monte-Carlo sweep for the gradient = many fewer calculations)
Bibliography


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Nick Denson and Mark Joshi, Fast and accurate Greeks for the LIBOR market model

Michael Giles, Monte Carlo evaluation of sensitivities in computational finance, 2012

John Hull, Options, Futures and other derivatives, 2012
Thank You very much for Your attention