Using Abs-Linearization to Determine Local Minima of Piecewise Linear Constrained Programs

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Table of Contents

Piecewise Linear Optimization Problems

Minimization Problem and Active Signature Method

Numerical Example

Summary and Outlook
Piecewise Linear Optimization Problems

\[
\min_{x \in \mathbb{R}^n} \quad y := \varphi(x)
\]
\[
\text{s.t.} \quad G(x) = 0
\]
\[
H(x) \leq 0,
\]
where

\[
\varphi : \mathbb{R}^n \to \mathbb{R}
\]
\[
G : \mathbb{R}^n \to \mathbb{R}^m
\]
\[
H : \mathbb{R}^n \to \mathbb{R}^p
\]
are all continuous and piecewise linear functions.
Piecewise Linear Optimization Problems

\[ \min_{x \in \mathbb{R}^n} \quad y := \varphi(x) \]

s.t. \quad G(x) = 0 \quad \text{and} \quad H(x) \leq 0,

where

- \( \varphi : \mathbb{R}^n \to \mathbb{R} \)
- \( G : \mathbb{R}^n \to \mathbb{R}^m \)
- \( H : \mathbb{R}^n \to \mathbb{R}^p \)

are all continuous and piecewise linear functions.

Figure:
\[ \varphi : \mathbb{R}^2 \to \mathbb{R}, (x_1, x_2) \mapsto 0.3|x_1| + ||x_1| + x_2| + 1 \]
Reformulate Functions in Abs-Linear Form

The general Abs-Linear Form for a given function $y$ and constraint functions:

$$Mz = c_z + Zx + L|z|, \quad y = c_y + a^\top x + b^\top |z|,$$

$$G(x, |z|) = g + Ax + B|z|, \quad H(x, |z|) = h + Cx + D|z|$$

▶ $x \in \mathbb{R}^n$ is the vector of independent variables
▶ $z \in \mathbb{R}^s$ the vector of switching variables
▶ $c_z \in \mathbb{R}^s$, $c_y \in \mathbb{R}$, $g \in \mathbb{R}^m$, $h \in \mathbb{R}^p$ are constant vectors
▶ $Z \in \mathbb{R}^{s \times n}$, $L, M \in \mathbb{R}^{s \times s}$ are matrices, where $L$ is strictly lower triangular and $M$ unit lower triangular
▶ $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{m \times s}$, $C \in \mathbb{R}^{p \times n}$, $D \in \mathbb{R}^{p \times s}$ are matrices
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Abs-Linear Form:

\[ Mz = c_z + Zx + L|z|, \]
\[ y = c_y + a^T x + b^T |z|, \]
\[ G(x, |z|) = g + Ax + B|z|, \]
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Abs-Linear Form:

\[ Mz = c_z + Zx + L|z|, \]
\[ y = c_y + a^\top x + b^\top |z|, \]
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Example:

\[ y = \varphi(x) = 0.3|x_1| + |x_1| + x_2| + 1, \]
\[ G(x) = |x_1| - 1, \]
\[ H(x) = -|x_1| - \frac{1}{2}|x_2| + 2, \]
Abs-Linear Form:

\[
Mz = c_z + Zx + L|z|, \\
y = c_y + a^\top x + b^\top |z|, \\
G(x, |z|) = g + Ax + B|z|, \\
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y = \varphi(x) = 0.3|x_1| + |x_1| + x_2 + 1, \\
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Example:

\[ y = \varphi(x) = 0.3|x_1| + ||x_1| + x_2| + 1, \]
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Decomposition of $\mathbb{R}^n$ into Polyhedrals

- For each $x \in \mathbb{R}^n$ define the **signature vector** and matrix

$$\sigma = \sigma(x) = \text{sgn}(z(x)) \in \{-1, 0, 1\}^s$$

and

$$\Sigma = \Sigma(x) = \text{diag}(\sigma(x))$$
Decomposition of $\mathbb{R}^n$ into Polyhedrals

- For each $x \in \mathbb{R}^n$ define the **signature vector** and matrix

  \[
  \sigma = \sigma(x) = \text{sgn}(z(x)) \in \{-1, 0, 1\}^s \quad \text{and} \quad \Sigma = \Sigma(x) = \text{diag}(\sigma(x))
  \]

- For a fixed $\sigma$ define **signature domains**

  \[
  P_\sigma = \{ x \in \mathbb{R}^n : \text{sgn}(z(x)) = \sigma \} \subset \bar{P}_\sigma = \{ x \in \mathbb{R}^n : \Sigma z(x) = |z(x)| \} 
  \]
Example: Decomposition of $\mathbb{R}^2$
Minimization Problem

For each fixed signature vector $\sigma \in \{-1, 0, 1\}^s$ one obtains the quadratic optimization problem

$$\min_{x,z} \quad a^T x + b^T \Sigma z + \frac{1}{2} x^T Q x$$

s.t. $M|\Sigma|z = c + Zx + L\Sigma z,$

$|\tilde{\Sigma}|z = 0,$

$\Sigma z \geq 0,$

$g + Ax + B|z| = 0$

$h +Cx + D|z| \leq 0$

where $|\tilde{\Sigma}| = I - |\Sigma|$ is the complementary orthogonal projection to $|\Sigma|$. 
Set up the System Matrix

Apply KKT theory to get the symmetric linear system

\[
\begin{bmatrix}
Q & 0 & Z^\top & A^\top & C^\top \Omega \\
0 & \Sigma L^\top & -\Sigma M^\top & \Sigma B^\top & \Sigma D^\top \Omega \\
Z & L\Sigma - M\Sigma & 0 & 0 & 0 \\
A & B\Sigma & 0 & 0 & 0 \\
\Omega C & \Omega D\Sigma & 0 & 0 & \bar{\Omega}
\end{bmatrix}
\begin{bmatrix}
x \\
z \\
\lambda \\
\delta \\
\omega
\end{bmatrix}
=
\begin{bmatrix}
a \\
\Sigma b \\
c \\
g \\
\Omega h
\end{bmatrix},
\]

where \( \Omega = \text{sgn}(\omega) \)
Set up the System Matrix

Apply KKT theory to get the symmetric linear system

\[
\begin{bmatrix}
Q & 0 & Z^T & A^T & C^T\Omega \\
0 & |\Sigma| & \Sigma L^T - |\Sigma|M^T & \Sigma B^T & \Sigma D^T\Omega \\
Z & L\Sigma - M|\Sigma| & 0 & 0 & 0 \\
A & B\Sigma & 0 & 0 & 0 \\
\Omega C & \Omega D\Sigma & 0 & 0 & \tilde{\Omega}
\end{bmatrix}
\begin{bmatrix}
x \\
z \\
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\end{bmatrix}
=
\begin{bmatrix}
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\Sigma b \\
c \\
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\Omega h
\end{bmatrix},
\]

where \( \Omega = \text{sgn}(\omega) \) and \( \tilde{\Omega} = I - \Omega \) as projection onto the active inequality constraints.
Set up the System Matrix

Apply KKT theory to get the symmetric linear system

\[
\begin{bmatrix}
Q & 0 & Z^T & A^T & C^T \Omega \\
0 & \Sigma L^T - |\Sigma|M^T & \Sigma B^T & \Sigma D^T \Omega \\
Z & L\Sigma - M|\Sigma| & 0 & 0 & 0 \\
A & B\Sigma & 0 & 0 & 0 \\
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\]

where \( \Omega = \text{sgn}(\omega) \) and \( \tilde{\Omega} = I - \Omega \) as projection onto the active inequality constraints.

Yields also the optimality condition

\[
0 \leq \mu^T |\tilde{\Sigma}| := b^T |\tilde{\Sigma}| + \lambda^T L|\tilde{\Sigma}| - \lambda^T M\tilde{\Sigma} + \delta^T B|\tilde{\Sigma}| + \omega^T D|\tilde{\Sigma}|,
\]

where \( \tilde{\Sigma} = I - |\Sigma| \).
Compute Step Length

- Make a step from current iterate $x^k$ in direction $x$
  - step length from switching variable

$$\beta^z = \inf_{1 \leq i \leq s} \left\{ \beta^z_i \equiv \frac{-z^k_i}{z_i - z^k_i} \left| z_i^k(z_i - z^k_i) < 0 \right\} \in [0, \infty]$$
Compute Step Length

- Make a step from current iterate $x^k$ in direction $x$
  - step length from switching variable

\[ \beta_j^z = \inf_{1 \leq i \leq s} \left\{ \beta_i^z \equiv \frac{-z_i^k}{z_i - z_i^k} \left| z_i^k(z_i - z_i^k) < 0 \right\} \in ]0, \infty] \]

- Compute new point via

\[ \tilde{x} = (1 - \beta_j^z)x^k + \beta_j^z x \]
\[ \tilde{z} = (1 - \beta_j^z)z^k + \beta_j^z z \]
Compute Step Length

- Make a step from current iterate $x^k$ in direction $x$
  - step length from switching variable
    \[
    \beta^z_j = \inf_{1 \leq i \leq s} \left\{ \beta^z_i \equiv \frac{-z^k_i}{z_i - z^k_i} \left| z^k_i (z_i - z^k_i) < 0 \right| \right\} \in ]0, \infty] 
    \]

- Compute new point via
  \[
  \tilde{x} = (1 - \beta^z_j) x^k + \beta^z_j x \\
  \tilde{z} = (1 - \beta^z_j) z^k + \beta^z_j z 
  \]

- Make a step from current iterate $x^k$ in direction $\tilde{x}$
  - step length from constraints
    \[
    \beta^H_j = \inf_{1 \leq i \leq p} \left\{ \beta^H_i \equiv \frac{H^k_i}{H^k_i - H_i(\tilde{x}, \Sigma \tilde{z})} \left| (H_i(\tilde{x}, \Sigma \tilde{z}) - H^k_i) H^k_i < 0 \right| \right\} . 
    \]
Algorithm:
Active Signature Method for Constrained Abs-Linear Minimization

Given: Problem in Abs-Linear Form, feasible starting point $x$ and associated $z$ and $\sigma$
Algorithm:
Active Signature Method for Constrained Abs-Linear Minimization

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Procedure: while optimality condition is violated do
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Procedure: while optimality condition is violated do

- Step 1: Set up and solve linear system of equations
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Active Signature Method for Constrained Abs-Linear Minimization

Given: Problem in Abs-Linear Form, feasible starting point $x$ and associated $z$ and $\sigma$

Procedure: while optimality condition is violated do

- Step 1: Set up and solve linear system of equations
- Step 2: Compute step length and if necessary add or drop kinks or constraints
Algorithm:
Active Signature Method for Constrained Abs-Linear Minimization

Given: Problem in Abs-Linear Form, feasible starting point $x$, and associated $z$ and $\sigma$

Procedure: while optimality condition is violated do
  ▶ Step 1: Set up and solve linear system of equations
  ▶ Step 2: Compute step length and if necessary add or drop kinks or constraints
until problem solved
Numerical Example

Problem:

\[ \min \ 0.3|x_1| + |x_1| + x_2 + 1 \]
Numerical Example

Problem:

$$\begin{align*}
\text{min} & \quad 0.3 |x_1| + |x_1| + x_2| + 1 \\
\text{s.t.} & \quad |x_1| + \frac{1}{2} |x_2| \geq 2
\end{align*}$$
Numerical Example

Problem:

\[ \begin{align*}
\min & \quad 0.3|x_1| + \frac{3}{2}|x_1 + x_2| + 1 \\
\text{s.t.} & \quad |x_1| + \frac{1}{2}|x_2| \geq 2 \\
& \quad |x_1| = 1
\end{align*} \]
Summary

- Consider optimization problems with piecewise linear objective function and piecewise linear constraints
- Rewrite a function in Abs-Linear Form
- Set up and solve linear system of equations

Outlook

- Prove convergence results
- Check convergence rate
- Extend approach to SALMIN (successive abs-linear minimization) to cover piecewise smooth functions
Summary

▶ Consider optimization problems with piecewise linear objective function and piecewise linear constraints
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