Vibroto and Automatic Differentiation for High Order Derivatives and Sensitivities of Financial Options

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General problem

Due to BASEL III regulations, banks are requested to evaluate the sensitivities of their portfolios every day. Portfolios are usually huge and sensitivities are time consuming to compute. Faced with the problem of building a software for this task and distorting automatic differentiation for non-differentiable functions, we developed a new method:

- Combines Vibrot (see Mike Giles in [1]) and AD
- Faster than standard methods
- Provides a generic formula
- Easy to implement

Vibrot and First Order Derivative

Vibrot is based on a reformulation of the payoff which is better suited to differentiation. The path is split into the last time step and its past.

1 Vibrot for European Contracts

Let X(t) = (X(t),κ) be a diffusion process, the strong solution of the following stochastic differential equation (SDE)

\[ dX_t = \theta(X_t) dt + \sigma(X_t) dW_t, \quad X_0 = x. \] (1)

Given an integer n > 0, the Euler scheme with constant step h = \frac{T}{n}, is recursively defined by, for k = 1, ..., n,

\[ X_{k+1} = X_k + \theta(X_k) h + \sigma(X_k) \sqrt{h} Z_k, \quad X_0 = x \] (2).

where \( (Z_k)_{k=1}^{\infty} \) are independent random Gaussian \( N(0,1) \) vectors. Note that \( X_{n+1} = \mu_{n+1}(\theta) + \sigma_{n+1}(\theta) \sqrt{h} Z_n \) with

\[ \mu_{n+1}(\theta) = \mu_n(\theta) + \theta(X_n) h, \quad \sigma_{n+1}(\theta) = \sigma(\theta, X_n) \sqrt{h}. \] (3)

Scheme of simulation path of the Vibrot decomposition.

Then, for any Borel function \( V : \mathbb{R} \to \mathbb{R} \) such that \( \mathbb{E}[V(X(T))] < +\infty \),

\[ \mathbb{E}[V(X(T))] = \mathbb{E}[\mathbb{E}[V(X(T)) | X_n]]. \] (4)

Furthermore, by homogeneity of the chain,

\[ \mathbb{E}[V(X(T)) | X_n] = \mathbb{E}[V(\mu + s\sqrt{T}) \mathbb{E}[\sqrt{T} | X_n]], \] (5)

Where \( X_t^n(\theta) \) denotes the value at time \( t^n \) of the Euler scheme with \( n \)-steps, starting at \( x \) and where the last expectation is with respect to \( Z \). To lighten the text, we will remove "\( n \)" from the following formulas.

2 First Order Vibrot

From (4) and (5)

\[ \frac{\partial}{\partial \theta} \mathbb{E}[V(X(T))] = \mathbb{E} \left[ \frac{\partial}{\partial \theta} \left( V(\mu + s\sqrt{T}) \mathbb{E}[\sqrt{T} | X_n] \right) \right]. \] (6)

High Order Derivative

Theorem 1. Second Order Derivative of Vibrot (VAD)

\[ \frac{\partial^2}{\partial \theta^2} \mathbb{E}[V(X(T))] = \mathbb{E} \left[ \frac{\partial^2}{\partial \theta^2} \left( V(\mu + s\sqrt{T}) \mathbb{E}[\sqrt{T} | X_n] \right) \right]. \] (7)

Remark 1. In [3], we can see that computing analytically the second order derivative from the first order either by differentiating or using a second times the Vibrot reformulation results in the same formula. Hence, we use the direct differentiation of Vibrot with AD.

3 Higher Order Vibrot

The Vibrot-AD method can be generalized to higher order of differentiation of Vibrot with respect to the parameter \( \theta \) with the help of the Faà di Bruno formula and its generalization to a composite function with a vector argument, as in Mishchuk [2].

Non-Differentiable Functions

4 Ramp function

The second derivative in \( N \) of \( f(z) = \frac{z}{z} \) does not exist at \( z = K \).

- Distribution theory extends the notion of derivative.
- The Heaviside function \( h(z) = 1_{\{z \geq 0\}} \) has the Dirac mass at zero \( 0 \) for derivative.
- AD can be extended to handle this difficulty to some degree by approximating the Dirac mass at \( 0 \) by the functions \( h^{(z)} \) defined by

\[ h^{(z)}(z) = \frac{e^{-\frac{1}{2}z}}{\sqrt{2\pi}}. \]

Suppose \( f \) is discontinuous at \( z = 0 \) and smooth elsewhere;

\[ f(z) = f \left( f'(z) H(z-1) + f'(z) H(z-1) - f(z) - f'(z) H(z-1) \right) \]

In the AD library, defines

- \( z^2 \) is defined as \( z^2 \) with its derivative to \( H(z) \)
- \( H(z) \) is defined with its derivative to \( \sqrt{z} \)

3. The asset should be written as \( f(z - K) - \text{ramp}(z - K) \)

Then the 2nd derivative in \( V \) via AD will be \( f^{(z)} \).

VAD and the Black-Scholes Model

Let us take the example of a standard European Call option in the Black-Scholes model.

5 Conceptual algorithm for VAD

1. Generate \( M \) simulation paths with time step \( h = \frac{T}{M} \) of the underlying asset \( X \) and its tangent process \( Y = \frac{\partial}{\partial \theta} \) with respect to a parameter \( \theta \) for \( k = 0, \ldots, n = 2 \):

\[ X_{n+1} = X_n + \sqrt{h} \sigma X_n Z, \quad X_0 = \lambda \] (8).

2. For each simulation path

(a) Generate \( M \) last time steps \( (X_n = X^i_n) \)

\[ X_{n+1} = X_n + (1 + h + \sigma^2 T) \] (9).

(b) Compute the first derivative with respect to \( \theta \)

\[ \frac{\partial V}{\partial \theta} = \frac{\partial V}{\partial x} \cdot \frac{Z}{\sqrt{T}} + \frac{\partial^2 V}{\partial x^2} \cdot \frac{1}{2} \frac{Z^2}{\sqrt{T}} \] (10).

With

\[ \frac{\partial V}{\partial \theta} = \frac{\partial V}{\partial x} \cdot \frac{Z}{\sqrt{T}} + \frac{\partial^2 V}{\partial x^2} \cdot \frac{1}{2} \frac{Z^2}{\sqrt{T}} \]

\[ \frac{\partial^2 V}{\partial \theta^2} = \frac{\partial^2 V}{\partial x^2} \cdot \frac{1}{2} \frac{Z^2}{\sqrt{T}} \] (11).

If \( T = 0 \) or \( \theta = 0 \), we have to add \( \frac{\partial V}{\partial \theta} \).

(c) Apply an AD method that implements step 10 to compute the second derivative with respect to \( \theta \) at some \( \theta^2 \).

(d) Compute the mean per path i.e. over \( M \) and 10.

3. Compute the mean of the resulting vector (over the \( M \) simulation paths) and discount it.

Numerical Results

6 European Option

6.1 Second Order Derivatives

We use the following parameters for the European call option:

\[ \sigma = 20\%, \quad \rho = 0\%, \quad T = 1 \text{ year}, \quad K = 100 \] (12)

On the left the Gamma versus Price is displayed when computed by VAD; the analytical exact Gamma is also displayed. On the right, the convergence history at one point \( X_0 = 120 \) is displayed.

6.2 Third Order Derivatives

For third order derivatives, we compute second derivatives by VAD analytically and differentiability by AD (VVAD). The sensitivity of the Gamma with respect to changes in \( X_0 \) is \( \frac{\partial^3 V}{\partial X_0^3} \).

VVAD and the Black-Scholes Model

Let us take the example of a standard European Call option in the Black-Scholes model.

7 American Option

Let \( \phi \) be the payoff, then

\[ V_t = \text{ess sup} \{ e^{-r(T-t)} \phi(X_t) | X_t \} \] (13)

where \( T \) denotes the set of \( [T'] \)-valued stopping times (with respect to the (augmented) filtration of the process \( (X_t)_{t \in [T]} \)).

Its value is defined recursively by the following BDPP

\[ V_t = e^{-r(T-t)} \phi(X_t) \] (14)

See more details in [3].

Convergence of the Gamma of an American option via VAD on the Longstaff-Schwartz algorithm and via Finite Difference, for the number of simulation paths. The parameters are \( s = 0.5\%, \quad X_0 = 40, \quad K = 40, \quad T = 1 \) and \( r = 0.5\% \).

References


[3] G. Pagès, D. Pironneau et G. Sell, Vibrot and automatic differentiation for High order derivative sensitivities and financial options, Available at: https://hal.archives-ouvertes.fr/hal-01244607

On the left the Vanna versus Price is displayed when computed by VVAD; the analytical exact Vanna is also displayed. On the right, the same for the Vanna with respect to changes in interest rate \( \frac{\partial^2 V}{\partial X_0^2} \).

On the left the \( \frac{\partial^2 V}{\partial X_0^2} \) versus Price is displayed when computed by VVAD; the analytical exact curve is also displayed. On the right, the same for the Vanna with respect to changes in interest rate \( \frac{\partial^2 V}{\partial X_0^2} \).